Some useful relations of the new star product on noncommutative curved space time when $\triangle^{\mu}$ is a constant

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We are going to show that new star product was found by Mebaraki for noncommutative curved space time has some properties when $\triangle$ (it has appeared in new multiplication) is a constant.

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Introduction

Recently, Mebaraki, Khallili, Bussahel and Haouchine derived the Moyal-Weyl star product for a noncommutative curved space time[1]. So it is a non associative product and it was introduced in following form

$$(f \triangleright g)(x) \quad = \quad e^{\triangle x^{\mu}\partial_{\mu}(\Im_f \otimes \Im_g)+\frac{\imath}{2}\theta^{\mu\nu}\partial_{\mu}\Im_f \otimes \partial_{\nu}\Im_g} f(x) \otimes g(y) \mid_{y=x} \tag{1}$$

Where $\triangle^{\mu} \equiv \frac{\imath^2}{2\sqrt{-g}}\theta^{\alpha\beta}\theta^{\delta\sigma}\partial_{\sigma}R^{\mu}_{\delta\alpha\beta}$ and "R" stands for the Reimann curvature tensor and $\Im$ means identity of spaces (for example $\Im_f$ stands for identity of "f" space) and so on. The physics will be approximate while the $\triangle x$ is not a constant or metric tensor as desired may be selected. the new star product is an approximate theory in terms of θ that B-C-H terms has its demands. So if we consider of $\triangle^{\mu} \equiv \frac{\imath^2}{2\sqrt{-g}}\theta^{\alpha\beta}\theta^{\delta\sigma}\partial_{\sigma}R^{\mu}_{\delta\alpha\beta}$ is a constant we get to exact theory and we will have a good behavior theory otherwise many concepts will change for example $\int\ dx\ f(x) \triangleright \delta(x-a) \neq f(a)$ and our theories will be very complex. However, this is an important class of non-commutativity for space time configurations with out curvature. Let's to derive the following formula in noncommutative flat space time ($\star$ stands for Moyal star product)

$$\int\ dx\ f(x) \star \delta(x-a) = \int\ dx\ f(x)\delta(x-a) = f(a) \tag{2}$$

But in continue, we show that this isn't true for noncommutative curved space time with general metric.

Non accosiated product and relations

We start the proof of following formula for flat space time

$$\int\ dx\ f(x) \star \delta(x-a) \star g(x) = g(x) \star f(x)_{|x=a} \tag{3}$$

This is our starting point $\mathbf{dk} \equiv \frac{1}{(2\pi)^{\frac{d}{2}}} d^d k$ and $dx \equiv d^d x$ so by the Moyal-Weyl map we get to (at this time, for convenient, we introduce $\S = f(x) \star \delta(x-a) \star g(x)$)

$$
\begin{aligned}
\S &= \int \mathbf{dkdqdp} \;\; \tilde{f}(k)\tilde{g}(p)e^{\imath k\hat{x}}e^{\imath q(\hat{x}-a)}e^{\imath p\hat{x}} \\
&= \int \mathbf{dkdqdp} \;\; \tilde{f}(k)\tilde{g}(p)e^{-\imath qa}e^{\imath(k+q)\hat{x}}e^{\imath p\hat{x}}e^{+\frac{1}{2}[\imath k_\mu\hat{x}^\mu \;,\; \imath q_\nu\hat{x}^\nu]} \\
&= \int \mathbf{dkdqdp} \; \tilde{f}(k)\tilde{g}(p)e^{-\imath qa}e^{\imath(k+q)\hat{x}}e^{\imath p\hat{x}}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}q_\nu}
\end{aligned} \tag{4}
$$

We introduce $\eta = k + q$ so we have

$$
\begin{aligned}
\S &= \int \mathbf{dkd}\eta\mathbf{dp} \;\; \tilde{f}(k)\tilde{g}(p)e^{-\imath(\eta-k)a}e^{\imath\eta\hat{x}}e^{\imath p\hat{x}}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(\eta-k)_\nu} \\
&= \int \mathbf{dkd}\eta\mathbf{dp} \;\; \tilde{f}(k)\tilde{g}(p)e^{-\imath(\eta-k)a}e^{\imath(\eta+p)\hat{x}}e^{\frac{-\imath}{2}\eta_\mu\theta^{\mu\nu}p_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(\eta-k)_\nu}
\end{aligned} \tag{5}
$$

Again, we introduce $\alpha = \eta + p$ so we have

$$
\begin{aligned}
\S &= \int \mathbf{dkd}\eta\mathbf{dp} \;\; \tilde{f}(k)\tilde{g}e^{-\imath(\eta-k)a}(p)e^{\imath(\eta+p)\hat{x}}e^{\frac{-\imath}{2}\eta_\mu\theta^{\mu\nu}p_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(\eta-k)_\nu} \\
&= \int \mathbf{dkd}\eta\mathbf{d}\alpha \;\; \tilde{f}(k)\tilde{g}(\alpha-\eta)e^{-\imath(\eta-k)a}e^{\imath\alpha\hat{x}}e^{\frac{-\imath}{2}\eta_\mu\theta^{\mu\nu}(\alpha-\eta)_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(\eta-k)_\nu}
\end{aligned} \tag{6}
$$

From the Moyal-Weyl map we get to

$$
W(\widetilde{f\star g}) = \int \;\; \mathbf{d}\alpha \; \widetilde{f\star g}(\alpha)e^{\imath\alpha\hat{x}} \tag{7}
$$

Where

$$
\widetilde{f\star g}(\alpha) = \int \mathbf{dkd}\eta\tilde{f}(k)\tilde{g}(\alpha-\eta)e^{-\imath(\eta-k)a}e^{\frac{-\imath}{2}\eta_\mu\theta^{\mu\nu}(\alpha-\eta)_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(\eta-k)_\nu} \tag{8}
$$

So we can write [2]

$$
f\star g(x) = \int \mathbf{d}\alpha(\int \mathbf{dkd}\eta\tilde{f}(k)\tilde{g}(\alpha-\eta)e^{-\imath(\eta-k)a}e^{\frac{-\imath}{2}\eta_\mu\theta^{\mu\nu}(\alpha-\eta)_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(\eta-k)_\nu})e^{\imath\alpha x} \tag{9}
$$

Now we get back to initials parameters $\alpha = \eta + p$

$$
\begin{aligned}
f \star g(x) &= \int \mathbf{dkd}\eta\mathbf{dp}\tilde{f}(k)\tilde{g}(p)e^{-\imath qa}e^{\frac{-\imath}{2}\eta_\mu\theta^{\mu\nu}p_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(\eta-k)_\nu}e^{\imath(\eta+p)x} \\
&= \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{g}(p)e^{-\imath qa}e^{\frac{-\imath}{2}(k+q)_\mu\theta^{\mu\nu}p_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(k+q-k)_\nu}e^{\imath(k+q+p)x} \\
&= \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{g}(p)e^{\frac{-\imath}{2}(k+q)_\mu\theta^{\mu\nu}p_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}q_\nu}e^{\imath(k+q+p)x}e^{-\imath qa}
\end{aligned}
\tag{10}
$$

So

$$
\begin{aligned}
& \int dx\ f(x) \star \delta(x-a) \star g(x) \\
&= \int dx \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{g}(p)e^{\frac{-\imath}{2}(k+q)_\mu\theta^{\mu\nu}p_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}q_\nu}e^{\imath(k+q+p)x}e^{-\imath qa} \\
&= \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{g}(p)e^{\frac{-\imath}{2}(k+q)_\mu\theta^{\mu\nu}p_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}q_\nu}(\int dx\ e^{\imath(k+q+p)x})e^{-\imath qa} \\
&= \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{g}(p)e^{\frac{-\imath}{2}(k+q)_\mu\theta^{\mu\nu}p_\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}q_\nu}(\delta(k+q+p))e^{-\imath qa} \\
&= \int \mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\frac{\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)a} \\
&= \int dx\ \delta(y-a)\mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\frac{\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)y} \\
&= g(y) \star f(y)_{y=a}
\end{aligned}
\tag{11}
$$

We use the our proven method for another formula in flat space time (at this time, $\S = \frac{\delta}{\delta h(z)}\int\ dx\ f(x) \star h(x) \star g(x))$

$$
\begin{aligned}
\S &= \frac{\delta}{\delta h(z)}\int dx\ \mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath k\hat{x}}e^{\imath q\hat{x}}e^{\imath p\hat{x}} \\
&= \frac{\delta}{\delta h(z)}\int dx\ \mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath(k+q+p)\hat{x}}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}q_\nu}e^{\frac{-\imath}{2}(k+q)_\mu\theta^{\mu\nu}p_\nu}
\end{aligned}
\tag{12}
$$

By the Moyal-Weyl map we get to

$$\begin{aligned}
\S &= \frac{\delta}{\delta h(z)}\int dx\,\mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath(k+q+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}q_\nu}e^{\frac{-\imath}{2}(k+q)_\mu\theta^{\mu\nu}p_\nu}\\
&= \frac{\delta}{\delta h(z)}\int dx\,\mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath(k+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath q_\nu x^\nu}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}q_\nu}e^{\frac{-\imath}{2}q_\mu\theta^{\mu\nu}p_\nu}\\
&= \frac{\delta}{\delta h(z)}\int dx\,\mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}\Big(\int\mathbf{dq}e^{\imath q_\mu(x^\mu-\frac{1}{2}k_\nu\theta^{\nu\mu}-\frac{1}{2}\theta^{\mu\nu}p_\nu)}\tilde{h}(q)\Big)\\
&= \int dx\,\mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}\delta((x^\mu-\frac{1}{2}k_\nu\theta^{\nu\mu}-\frac{1}{2}\theta^{\mu\nu}p_\nu)-z^\mu)\\
&= \int\mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)_\mu(z^\mu+\frac{1}{2}k_\nu\theta^{\nu\mu}+\frac{1}{2}\theta^{\mu\nu}p_\nu)}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}\\
&= g(y)\star f(y)\,|_{y=z}
\end{aligned}\tag{13}$$

At this time, we try to construct the similar formulas for the non-associative new star product. So we can write (again, we introduce $\S = f(x)\triangleright(h(x)\triangleright g(x))$)

$$\begin{aligned}
\S &= W(f)(W(h)W(g)) = \int\mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath k\hat{x}}(e^{\imath q\hat{x}}e^{\imath p\hat{x}})\\
&= \int\mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath k\hat{x}}(e^{\imath(q+p)\hat{x}}e^{\frac{-\imath}{2}q_\mu\theta^{\mu\nu}p_\nu}e^{\imath(q+p)_\mu\triangle^\mu})
\end{aligned}\tag{14}$$

we introduce $\eta = q+p$ so we can write

$$\begin{aligned}
\S &= \int\mathbf{dkd\eta dp}\tilde{f}(k)\tilde{h}(\eta-p)\tilde{g}(p)e^{\imath k\hat{x}}(e^{\imath\eta\hat{x}}e^{\frac{-\imath}{2}(\eta-p)_\mu\theta^{\mu\nu}p_\nu}e^{\imath\eta_\mu\triangle^\mu})\\
&= \int\mathbf{dkd\eta dp}\tilde{f}(k)\tilde{h}(\eta-p)\tilde{g}(p)e^{\imath(k+\eta)\hat{x}}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}\eta_\nu}e^{\imath(k+\eta)_\mu\triangle^\mu}e^{\frac{-\imath}{2}(\eta-p)_\mu\theta^{\mu\nu}p_\nu}e^{\imath\eta_\mu\triangle^\mu}
\end{aligned}\tag{15}$$

We introduce $\alpha = k+q+p$

$$\begin{aligned}
\S &= \int\mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath(k+q+p)\hat{x}}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(q+p)_\nu}e^{\imath(k+q+p)_\mu\triangle^\mu}e^{\frac{-\imath}{2}q_\mu\theta^{\mu\nu}p_\nu}e^{\imath(q+p)_\mu\triangle^\mu}\\
&= \int\mathbf{d\alpha d\eta dp}\tilde{f}(\alpha-\eta)\tilde{h}(\eta-p)\tilde{g}(p)e^{\imath\alpha\hat{x}}e^{\frac{-\imath}{2}(\alpha-\eta)_\mu\theta^{\mu\nu}\eta_\nu}e^{\imath\alpha_\mu\triangle^\mu}e^{\frac{-\imath}{2}(\eta-p)_\mu\theta^{\mu\nu}p_\nu}e^{\imath\eta_\mu\triangle^\mu}\\
&= \int\mathbf{d\alpha}\Re(\alpha)e^{\imath\alpha\hat{x}}
\end{aligned}\tag{16}$$

Where

$$\Re(\alpha) = \mathbf{d\eta dp}\tilde{f}(\alpha-\eta)\tilde{h}(\eta-p)\tilde{g}(p)e^{\frac{-\imath}{2}(\alpha-\eta)_\mu\theta^{\mu\nu}\eta_\nu}e^{\imath\alpha_\mu\triangle^\mu}e^{\frac{-\imath}{2}(\eta-p)_\mu\theta^{\mu\nu}p_\nu}e^{\imath\eta_\mu\triangle^\mu}\tag{17}$$

So

$$\S = \int \mathbf{d}\alpha\mathbf{d}\eta\mathbf{dp}\tilde{f}(\alpha-\eta)\tilde{h}(\eta-p)\tilde{g}(p)e^{\imath\alpha x}e^{\frac{-\imath}{2}(\alpha-\eta)_\mu\theta^{\mu\nu}\eta_\nu}e^{\imath\alpha_\mu\triangle^\mu}e^{\frac{-\imath}{2}(\eta-p)_\mu\theta^{\mu\nu}p_\nu}e^{\imath\eta_\mu\triangle^\mu} \tag{18}$$

Now we get back to initials parameters

$$\begin{aligned}\int dx\ \S(x) &= \int dx \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p) \\ &\quad e^{\imath(k+q+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}(q+p)_\nu}e^{\imath(k+q+p)_\mu\triangle^\mu}e^{\frac{-\imath}{2}q_\mu\theta^{\mu\nu}p_\nu}e^{\imath(q+p)_\mu\triangle^\mu} \\ &= \int dx\mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)_\mu\triangle^\mu}e^{\imath p_\mu\triangle^\mu} \\ &\quad (\int \mathbf{dq}\ \tilde{h}(q)e^{\imath q_\mu(x^\mu+\frac{1}{2}\theta^{\mu\nu}k_\nu+2\triangle^\mu-\frac{1}{2}\theta^{\mu\nu}p_\nu)}) \\ &= \int dx\mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)_\mu\triangle^\mu}e^{\imath p_\mu\triangle^\mu}h(\bar{x})\end{aligned} \tag{19}$$

Where $\bar{x}^\mu = x^\mu + \frac{1}{2}\theta^{\mu\nu}k_\nu + 2\triangle^\mu - \frac{1}{2}\theta^{\mu\nu}p_\nu$ so we can write

$$\begin{aligned}\frac{\delta}{\delta h(z)}\int dx\ \S(x) &= \int dx\mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)_\mu\triangle^\mu}e^{\imath p_\mu\triangle^\mu}\frac{\delta}{\delta h(z)}h(\bar{x}) \\ &= \int dx\mathbf{dkdp}\delta(\bar{x}-z)\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)x}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)_\mu\triangle^\mu}e^{\imath p_\mu\triangle^\mu} \\ &= \int \mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)\bar{z}}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)_\mu\triangle(\bar{z})^\mu}e^{\imath p_\mu\triangle(\bar{z})^\mu}\end{aligned} \tag{20}$$

For first class of $\triangle x$ we can write $x^\mu + 2\triangle^\mu \equiv s(x^\mu)$ or $s(x^\mu) + \frac{1}{2}\theta^{\mu\nu}k_\nu - \frac{1}{2}\theta^{\mu\nu}p_\nu = z^\mu$ then $s(x^\mu) = z^\mu - \frac{1}{2}\theta^{\mu\nu}k_\nu + \frac{1}{2}\theta^{\mu\nu}p_\nu$ if $\exists s^{-1}$ then $x^\mu = s^{-1}(z^\mu - \frac{1}{2}\theta^{\mu\nu}k_\nu + \frac{1}{2}\theta^{\mu\nu}p_\nu)$ so

$$\frac{\delta}{\delta h(z)}\int dx\ \S(x) = \int \mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)s^{-1}(\bar{z})}e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)_\mu\triangle^\mu(s^{-1}(\bar{z}))}e^{\imath p_\mu\triangle^\mu(s^{-1}(\bar{z}))} \tag{21}$$

Where $\bar{z}^\mu = z^\mu - \frac{1}{2}\theta^{\mu\nu}k_\nu + \frac{1}{2}\theta^{\mu\nu}p_\nu$. But for second class of $\triangle x$ ($\triangle x$ is a constant)

we have $x^\mu = z^\mu - 2\triangle^\mu - \frac{1}{2}\theta^{\mu\nu}k_\nu + \frac{1}{2}\theta^{\mu\nu}p_\nu$ so

$$\begin{aligned}
\frac{\delta}{\delta h(z)}\int dx\ \S(x) &= \int \mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)_\mu(z^\mu - 2\triangle^\mu - \frac{1}{2}\theta^{\mu\nu}k_\nu + \frac{1}{2}\theta^{\mu\nu}p_\nu)} \\
&\quad e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)_\mu\triangle^\mu}e^{\imath p_\mu\triangle^\mu} \\
&= \int \mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)_\mu z^\mu}e^{-2\imath(k+p)_\mu\triangle^\mu}e^{-\imath(k+p)_\mu\frac{1}{2}\theta^{\mu\nu}k_\nu}e^{+\imath(k+p)_\mu\frac{1}{2}\theta^{\mu\nu}p_\nu} \\
&\quad e^{\frac{-\imath}{2}k_\mu\theta^{\mu\nu}p_\nu}e^{\imath(k+p)_\mu\triangle^\mu}e^{\imath p_\mu\triangle^\mu} \\
&= \int dy\ \delta(y-z)\mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)_\mu y^\mu}e^{-\imath k_\mu\triangle^\mu} \\
&\quad e^{-\imath p_\mu\frac{1}{2}\theta^{\mu\nu}k_\nu} \\
&= g(y)\triangleright_1 f(y)\,|_{y=z} \qquad (22)
\end{aligned}$$

Where $\triangleright_1 = e^{\frac{\imath}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\otimes\overrightarrow{\partial}_\nu - \triangle^\mu\overleftarrow{\Im}\otimes\overrightarrow{\partial}_\mu}$ and by similar way we can find

$$\frac{\delta}{\delta h(z)}\int dx\ f\triangleright(g\triangleright h) = f(y)\triangleleft_1 g(y)\,|_{y=z} \qquad (23)$$

Where $\triangleleft_1 = e^{\frac{\imath}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\otimes\overrightarrow{\partial}_\nu - \triangle^\mu\overleftarrow{\partial}_\mu\otimes\overrightarrow{\Im}}$. Also we can write

$$\int dx\ f\triangleright\delta(x-z) = \int dx\ \delta(x-z)\triangleright f(x) = f(z) \qquad (24)$$

And for another formula we can write ($\S = f(x)\triangleright(h(x)g(x))$) for calculate of $\frac{\delta}{\delta h(z)}\S$
We have

$$\begin{aligned}
h(x)g(x) &= \int \mathbf{dqdp}\tilde{h}(q)\tilde{g}(p)e^{\imath qx}e^{\imath px} \\
&= \int \mathbf{d}\eta\mathbf{dp}\tilde{h}(\eta-p)\tilde{g}(p)e^{\imath\eta x} = \int \mathbf{d}\eta\tilde{H}(\eta)e^{\imath\eta x} \qquad (25)
\end{aligned}$$

Also $W(h(x)g(x)) = \int \mathbf{d}\eta\tilde{H}(\eta)e^{\imath\eta\hat{x}}$ where $\tilde{H}(\eta) = \int \mathbf{dp}\tilde{h}(\eta-p)\tilde{g}(p)$ so we can write

$$\begin{aligned}
\S &= \int dx \int \mathbf{dk d}\eta\tilde{f}(k)\tilde{H}(\eta)e^{\imath k\hat{x}}e^{\imath\eta\hat{x}} \\
&= \int dx \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath k\hat{x}}e^{\imath(q+p)\hat{x}} \\
&= \int dx \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath(k+q+p)\hat{x}}e^{\frac{-\imath}{2}\theta^{\mu\nu}k_\mu(q+p)_\nu}e^{\imath(k+q+p)_\mu\triangle^\mu} \qquad (26)
\end{aligned}$$

With the Moyal-Weyl map

$$
\begin{aligned}
\S &= \int dx \int \mathbf{dkdqdp}\tilde{f}(k)\tilde{h}(q)\tilde{g}(p)e^{\imath(k+q+p)x}e^{\frac{-\imath}{2}\theta^{\mu\nu}k_\mu(q+p)_\nu}e^{\imath(k+q+p)_\mu\triangle^\mu} \\
&= \int dx \int \mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)x}e^{-\frac{\imath}{2}\theta^{\mu\nu}k_\mu p_\nu}e^{\imath(k+p)_\mu\triangle^\mu} \\
&\quad (\mathbf{dq}\tilde{h}(q)e^{\imath q_\mu(x^\mu-\frac{1}{2}\theta^{\nu\mu}k_\nu+\triangle^\mu)}) \\
&= \int dx \int \mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)x}e^{\frac{-\imath}{2}\theta^{\mu\nu}k_\mu p_\nu}e^{\imath(k+p)_\mu\triangle^\mu}h(\bar{x})
\end{aligned}
\tag{27}
$$

So we get to

$$
\begin{aligned}
\frac{\delta}{\delta h(z)}\S &= \int\int \mathbf{dkdp}\tilde{f}(k)\tilde{g}(p)e^{\imath(k+p)_\mu(z^\mu+\frac{1}{2}\theta^{\nu\mu}k_\nu-\triangle^\mu)}e^{-\frac{\imath}{2}\theta^{\mu\nu}k_\mu p_\nu}e^{\imath(k+p)_\mu\triangle^\mu} \\
&= f(y)g(y)_{y=z}
\end{aligned}
\tag{28}
$$

And for next formula ($\S = \frac{\delta}{\delta h(z)}\int\ dx\ f(h \triangleright g)$)

$$
\begin{aligned}
\S &= \frac{\delta}{\delta h(z)}\int dx \int \mathbf{dp}\tilde{f}(p)e^{\imath px}(\int \mathbf{dqdk}\tilde{h}(q)\tilde{g}(k)e^{\imath q\hat{x}}e^{\imath k\hat{x}}) \\
&= \frac{\delta}{\delta h(z)}\int dx \int \mathbf{dp}\tilde{f}(p)e^{\imath px}(\int \mathbf{dqdk}\tilde{h}(q)\tilde{g}(k)e^{\imath(k+q)x}e^{\frac{-\imath}{2}q\theta k}e^{\imath(k+q)\triangle}) \\
&= \frac{\delta}{\delta h(z)}\int dx \int \mathbf{dpdk}\tilde{f}(p)\tilde{g}(k)e^{\imath(p+k)x}e^{\imath k_\mu\triangle^\mu}(\int \mathbf{dq}\tilde{h}(q)e^{\imath q_\mu(\frac{-1}{2}\theta^{\mu\nu}k_\nu+x^\mu+\triangle^\mu)}) \\
&= \int dx \int \mathbf{dpdk}\tilde{f}(p)\tilde{g}(k)e^{\imath(p+k)x}e^{\imath k_\mu\triangle^\mu}(\delta(x^\mu-(z^\mu-\triangle^\mu-\frac{-1}{2}\theta^{\mu\nu}k_\nu))) \\
&= \int \mathbf{dpdk}\tilde{f}(p)\tilde{g}(k)e^{\imath(p+k)(z^\mu-\triangle^\mu-\frac{-1}{2}\theta^{\mu\nu}k_\nu)}e^{\imath k_\mu\triangle^\mu} \\
&= g(z)e^{\frac{\imath}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\otimes\overrightarrow{\partial_\nu}-\triangle^\mu\overleftarrow{\Im}\otimes\overrightarrow{\partial}_\mu}f(z) = g(z)\triangleright_1 f(z)
\end{aligned}
\tag{29}
$$

And for $\S = \frac{\delta}{\delta h(z)}\int\ dx\ f(g \triangleright h)$ we can write

$$
\begin{aligned}
\S &= \frac{\delta}{\delta h(z)}\int dx \int \mathbf{dp}\tilde{f}(p)e^{\imath px}(\int \mathbf{dqdk}\tilde{h}(q)\tilde{g}(k)e^{\imath k\hat{x}}e^{\imath q\hat{x}}) \\
&= \frac{\delta}{\delta h(z)}\int dx \int \mathbf{dp}\tilde{f}(p)e^{\imath px}(\int \mathbf{dqdk}\tilde{h}(q)\tilde{g}(k)e^{\imath(k+q)x}e^{\frac{-\imath}{2}k\theta q}e^{\imath(k+q)\triangle}) \\
&= \frac{\delta}{\delta h(z)}\int dx \int \mathbf{dpdk}\tilde{f}(p)\tilde{g}(k)e^{\imath(p+k)x}e^{\imath k_\mu\triangle^\mu}(\int \mathbf{dq}\tilde{h}(q)e^{\imath q_\mu(\frac{-1}{2}\theta^{\nu\mu}k_\nu+x^\mu+\triangle^\mu)}) \\
&= \int dx \int \mathbf{dpdk}\tilde{f}(p)\tilde{g}(k)e^{\imath(p+k)x}e^{\imath k_\mu\triangle^\mu}(\delta(x^\mu-(z^\mu-\triangle^\mu-\frac{-1}{2}\theta^{\nu\mu}k_\nu))) \\
&= \int \mathbf{dpdk}\tilde{f}(p)\tilde{g}(k)e^{\imath(p+k)(z^\mu-\triangle^\mu-\frac{-1}{2}\theta^{\nu\mu}k_\nu)}e^{\imath k_\mu\triangle^\mu} \\
&= f(z)e^{\frac{\imath}{2}\theta^{\mu\nu}\overleftarrow{\partial_\mu}\otimes\overrightarrow{\partial_\nu}-\triangle^\mu\overleftarrow{\partial}_\mu\otimes\overrightarrow{\Im}}g(z) = f(z)\triangleleft_1 g(z)
\end{aligned}
\tag{30}
$$

Also we see that if ($\S = \mathbf{A}(\partial_\mu\mathbf{B}\triangleright\mathbf{C})$) we can write

$$
\frac{\delta}{\delta B(z)}\S =: \frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}(\partial_\mu\mathbf{B}\triangleright\mathbf{C})
\tag{31}
$$

But we have

$$
\begin{aligned}
\S &= \int dx\mathbf{dqdpdk}\ \imath q^\mu\tilde{A}(k)\tilde{B}(q)\tilde{C}(p)e^{\imath kx}e^{\imath(q+p)\hat{x}}e^{\frac{-\imath}{2}q\theta p}e^{\imath(p+q)\triangle} \\
&= -\int dx\int \mathbf{dpdk}\ \partial_\mu(\tilde{A}(k)\tilde{C}(p)e^{\imath kx}e^{\imath px}e^{\imath p\triangle})\int \mathbf{dq}\tilde{B}(q)e^{\imath q(x+\frac{-1}{2}\theta p+\triangle)}
\end{aligned}
\tag{32}
$$

Or

$$
\begin{aligned}
\frac{\delta}{\delta B(z)}\S &= -\int dx \int\mathbf{dpdk}\ \partial_\mu(\tilde{A}(k)\tilde{C}(p)e^{\imath kx}e^{\imath px}e^{\imath p\triangle})\delta(z-(x+\frac{-1}{2}\theta p+\triangle)) \\
&= -\int\mathbf{dpdk}\ \partial_\mu(\tilde{A}(k)\tilde{C}(p)e^{\imath kx}e^{\imath px}e^{\imath p\triangle})\ |_{x^\nu=z^\nu+\frac{1}{2}\theta^{\nu\alpha}p_\alpha-\triangle^\nu} \\
&= -\partial_\mu\int\mathbf{dpdk}\ \tilde{A}(k)\tilde{C}(p)e^{\imath kz}e^{\imath pz}e^{-\imath k\triangle+\frac{\imath}{2}k_\alpha\theta^{\alpha\nu}p_\nu} \\
&= -\partial_\mu(C(z)e^{-\triangle^\alpha\overleftarrow{\Im}\otimes\overrightarrow{\partial_\alpha}+\frac{\imath}{2}\theta^{\alpha\nu}\overleftarrow{\partial_\alpha}\otimes\overrightarrow{\partial_\nu}}A(z))
\end{aligned}
\tag{33}
$$

For new equation ($\S = \mathbf{A}(\mathbf{C}\triangleright\partial_\mu\mathbf{B})$) and for

$$
\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}(\mathbf{C}\triangleright\partial_\mu\mathbf{B})
\tag{34}
$$

We can write

$$\S \;=\; \int dx\mathbf{dqdpdk}\; \imath q^{\mu}\tilde{A}(k)\tilde{B}(q)\tilde{C}(p)e^{\imath kx}e^{\imath(q+p)\hat{x}}e^{\frac{-\imath}{2}p\theta q}e^{\imath(p+q)\triangle} \tag{35}$$

But this is sufficient to convert of $\theta \to -\theta$ in last relation, so we have

$$\frac{\delta}{\delta B(z)}\S \;=\; -\partial_{\mu}(A(z)e^{-\triangle^{\alpha}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\Im}+\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}C(z)) \tag{36}$$

And

$$\frac{\delta}{\delta B(z)}\int dx\; \mathbf{A}((\mathbf{B}\triangleright\mathbf{C})\triangleright\mathbf{D}) \;=\; C(z)e^{\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}(D(z)e^{-\triangle^{\alpha}\overleftarrow{\partial_{\alpha}}\otimes 2\overrightarrow{\partial_{\alpha}}+\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}A(z)) \tag{37}$$

And

$$\frac{\delta}{\delta B(z)}\int dx\; \mathbf{A}((\mathbf{C}\triangleright\mathbf{B})\triangleright\mathbf{D}) \;=\; (D(z)e^{-\triangle^{\alpha}\overleftarrow{\partial_{\alpha}}\otimes 2\overrightarrow{\partial_{\alpha}}+\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}A(z))e^{\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}C(z) \tag{38}$$

And

$$\frac{\delta}{\delta B(z)}\int dx\; \mathbf{A}(\mathbf{D}\triangleright(\mathbf{B}\triangleright\mathbf{C})) \;=\; C(z)e^{\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}(A(z)e^{-\triangle^{\alpha}\overleftarrow{2\partial_{\alpha}}\otimes\overrightarrow{\partial_{\alpha}}+\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}D(z)) \tag{39}$$

And

$$\frac{\delta}{\delta B(z)}\int dx\; \mathbf{A}(\mathbf{D}\triangleright(\mathbf{C}\triangleright\mathbf{B})) \;=\; (A(z)e^{-\triangle^{\alpha}\overleftarrow{2\partial_{\alpha}}\otimes\overrightarrow{\partial_{\alpha}}+\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}D(z))e^{\imath\frac{1}{2}\theta^{\alpha\nu}\overleftarrow{\partial_{\alpha}}\otimes\overrightarrow{\partial_{\nu}}}C(z) \tag{40}$$

By new symbols we have

$$\begin{aligned}
&Ae^{\imath\frac{1}{2}\theta^{\mu\nu}\overleftarrow{\partial_{\mu}}\otimes\overrightarrow{\partial_{\nu}}-\triangle^{\mu}\overleftarrow{j\partial_{\mu}}\otimes\overrightarrow{\Im}}B =: \triangleleft_{jA}\\
&Be^{\imath\frac{1}{2}\theta^{\mu\nu}\overleftarrow{\partial_{\mu}}\otimes\overrightarrow{\partial_{\nu}}-\triangle^{\mu}\overleftarrow{\Im}\otimes\overrightarrow{j\partial_{\mu}}}A =: \triangleright_{jA}\\
&e^{\imath\frac{1}{2}\theta^{\mu\nu}\overleftarrow{\partial_{\mu}}\otimes\overrightarrow{\partial_{\nu}}} =: \star
\end{aligned} \tag{41}$$

Or

$$
\begin{aligned}
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A} \triangleright (\mathbf{B} \triangleright \mathbf{C}) = C(z) \triangleright_{1A} A(z)\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A} \triangleright (\mathbf{C} \triangleright \mathbf{B}) = A(z) \triangleright_{1A} C(z)\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A} \triangleright (\mathbf{B}\mathbf{C}) = A(z)C(z)\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}(\mathbf{B} \triangleright \mathbf{C}) = C(z) \triangleright_{1A} A(z)\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}(\mathbf{C} \triangleright \mathbf{B}) = A(z) \triangleright_{1A} C(z)\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}(\partial_\mu \mathbf{B} \triangleright \mathbf{C}) = -\partial_\mu (C(z) \triangleright_{1A} A(z))\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}(\mathbf{C} \triangleright \partial_\mu \mathbf{B}) = -\partial_\mu (A(z) \triangleleft_{1A} C(z))\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}((\mathbf{B} \triangleright \mathbf{C}) \triangleright \mathbf{D}) = C(z) \star (D(z) \triangleright_{2A} A(z))\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}((\mathbf{C} \triangleright \mathbf{B}) \triangleright \mathbf{D}) = (D(z) \triangleright_{2A} A(z)) \star C(z)\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}(\mathbf{D} \triangleright (\mathbf{B} \triangleright \mathbf{C})) = C(z) \star (A(z) \triangleleft_{2A} D(z))\\
&\frac{\delta}{\delta B(z)}\int dx\ \mathbf{A}(\mathbf{D} \triangleright (\mathbf{C} \triangleright \mathbf{B})) = (A(z) \triangleleft_{2A} D(z)) \star C(z)
\end{aligned} \tag{42}
$$

And we will show it is (the new star product) a non-associative product much as Δx is a constant.

$$
f \triangleright (h \triangleright g) \ = \ \int \mathbf{dkdqdp} \tilde{f}(k)\tilde{h}(q)\tilde{g}(p) e^{\imath(k+p+q)\hat{x}} e^{\frac{-\imath}{2} k\theta(q+p)} e^{\imath(k+p+q)\triangle} e^{\frac{-\imath}{2} q\theta p} e^{\imath(p+q)\triangle} \tag{43}
$$

This is not same with

$$
(f \triangleright h) \triangleright g \ = \ \int \mathbf{dkdqdp} \tilde{f}(k)\tilde{h}(q)\tilde{g}(p) e^{\imath(k+p+q)\hat{x}} e^{\frac{-\imath}{2} (k+q)\theta p} e^{\imath(k+p+q)\triangle} e^{\frac{-\imath}{2} k\theta q} e^{\imath(k+q)\triangle} \tag{44}
$$

We see that these are not same.

Discussion

Formulas ware derived are useful when we work with new star product with special metric($\triangle^{\mu}$ will be a constant). We have shown that they have different results than flat space time. This is important to calculate of laws of physics such as variational method in field theory and electrodynamics. We are sure that recent results will lead us to the old results while $lim_{\triangle\to 0}$.

References

[1] N. Mebaraki, F. Khallili, M. Boussahel and M. Haouchine, " Modified Moyal-Weyl star product in a curved non commutative Space-Time", Electron.J.Theor.Phys.3:37,2006.

[2] J. Madore, S. Schraml, P. Schupp, J. Wess, "Gauge Theory on Noncommutative Spaces", hep-th/0001203.